

Classification

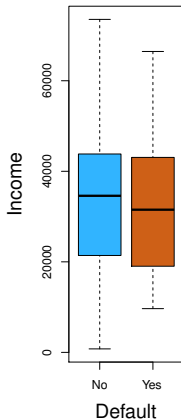
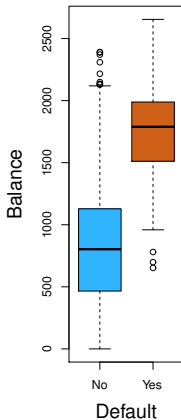
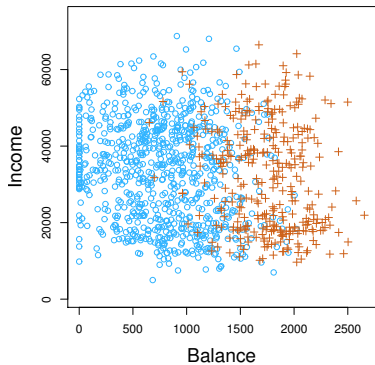
- Qualitative variables take values in an unordered set \mathcal{C} , such as:
 $\text{eye color} \in \{\text{brown, blue, green}\}$
 $\text{email} \in \{\text{spam, ham}\}$.
- Given a feature vector X and a qualitative response Y taking values in the set \mathcal{C} , the classification task is to build a function $C(X)$ that takes as input the feature vector X and predicts its value for Y ; i.e. $C(X) \in \mathcal{C}$.
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For example, it is more valuable to have an estimate of the probability that an insurance claim is fraudulent, than a classification fraudulent or not.

Example: Credit Card Default



Can we use Linear Regression?

Suppose for the **Default** classification task that we code

$$Y = \begin{cases} 0 & \text{if No} \\ 1 & \text{if Yes.} \end{cases}$$

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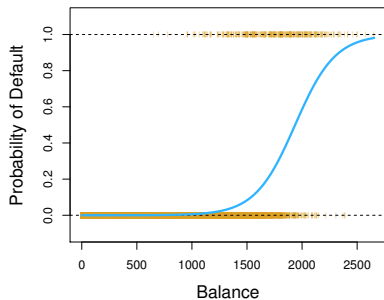
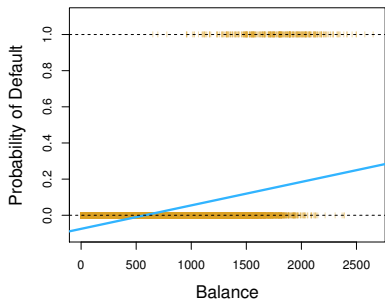
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- Since in the population $E(Y|X = x) = \Pr(Y = 1|X = x)$, we might think that regression is perfect for this task.
- However, *linear* regression might produce probabilities less than zero or bigger than one. *Logistic regression* is more appropriate.

Linear versus Logistic Regression



The orange marks indicate the response Y , either 0 or 1. Linear regression does not estimate $\Pr(Y = 1|X)$ well. Logistic regression seems well suited to the task.

Linear Regression continued

Now suppose we have a response variable with three possible values. A patient presents at the emergency room, and we must classify them according to their symptoms.

$$Y = \begin{cases} 1 & \text{if stroke;} \\ 2 & \text{if drug overdose;} \\ 3 & \text{if epileptic seizure.} \end{cases}$$

This coding suggests an ordering, and in fact implies that the difference between **stroke** and **drug overdose** is the same as between **drug overdose** and **epileptic seizure**.

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Linear regression is not appropriate here.

Multiclass Logistic Regression or *Discriminant Analysis* are more appropriate.

Logistic Regression

Let's write $p(X) = \Pr(Y = 1|X)$ for short and consider using **balance** to predict **default**. Logistic regression uses the form

$$p(X) = \frac{e^{\beta_0 + \beta_1 X}}{1 + e^{\beta_0 + \beta_1 X}}.$$

($e \approx 2.71828$ is a mathematical constant [Euler's number].)

It is easy to see that no matter what values β_0 , β_1 or X take, $p(X)$ will have values between 0 and 1.

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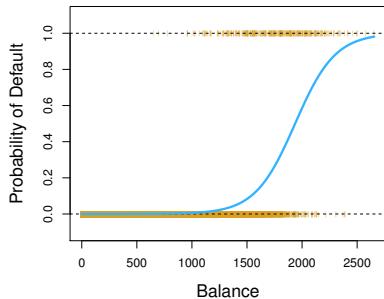
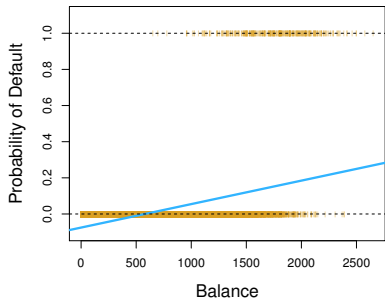
It is easy to see that no matter what values β_0 , β_1 or X take, $p(X)$ will have values between 0 and 1.

A bit of rearrangement gives

$$\log \left(\frac{p(X)}{1 - p(X)} \right) = \beta_0 + \beta_1 X.$$

This monotone transformation is called the *log odds* or *logit* transformation of $p(X)$. (by log we mean *natural log*: \ln .)

Linear versus Logistic Regression



Logistic regression ensures that our estimate for $p(X)$ lies between 0 and 1.

Maximum Likelihood

We use maximum likelihood to estimate the parameters.

$$\ell(\beta_0, \beta) = \prod_{i:y_i=1} p(x_i) \prod_{i:y_i=0} (1 - p(x_i)).$$

This *likelihood* gives the probability of the observed zeros and ones in the data. We pick β_0 and β_1 to maximize the likelihood of the observed data.

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Most statistical packages can fit linear logistic regression models by maximum likelihood. In **R** we use the `glm` function.

	Coefficient	Std. Error	Z-statistic	P-value
Intercept	-10.6513	0.3612	-29.5	< 0.0001
balance	0.0055	0.0002	24.9	< 0.0001

Making Predictions

What is our estimated probability of **default** for someone with a balance of \$1000?

$$\hat{p}(X) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 X}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 X}} = \frac{e^{-10.6513 + 0.0055 \times 1000}}{1 + e^{-10.6513 + 0.0055 \times 1000}} = 0.006$$

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With a balance of \$2000?

$$\hat{p}(X) = \frac{e^{\hat{\beta}_0 + \hat{\beta}_1 X}}{1 + e^{\hat{\beta}_0 + \hat{\beta}_1 X}} = \frac{e^{-10.6513 + 0.0055 \times 2000}}{1 + e^{-10.6513 + 0.0055 \times 2000}} = 0.586$$

Lets do it again, using **student** as the predictor.

	Coefficient	Std. Error	Z-statistic	P-value
Intercept	-3.5041	0.0707	-49.55	< 0.0001
student [Yes]	0.4049	0.1150	3.52	0.0004

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$$\widehat{\Pr}(\text{default}=\text{Yes}|\text{student}=\text{Yes}) = \frac{e^{-3.5041+0.4049 \times 1}}{1 + e^{-3.5041+0.4049 \times 1}} = 0.0431,$$

$$\widehat{\Pr}(\text{default}=\text{Yes}|\text{student}=\text{No}) = \frac{e^{-3.5041+0.4049 \times 0}}{1 + e^{-3.5041+0.4049 \times 0}} = 0.0292.$$

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Linear regression is not appropriate here.

Multiclass Logistic Regression or *Discriminant Analysis* are more appropriate.

Logistic regression with more than two classes

So far we have discussed logistic regression with two classes. It is easily generalized to more than two classes. One version (used in the R package `glmnet`) has the symmetric form

$$\Pr(Y = k|X) = \frac{e^{\beta_{0k} + \beta_{1k}X_1 + \dots + \beta_{pk}X_p}}{\sum_{\ell=1}^K e^{\beta_{0\ell} + \beta_{1\ell}X_1 + \dots + \beta_{p\ell}X_p}}$$

Here there is a linear function for *each* class.

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Multiclass logistic regression is also referred to as *multinomial regression*.

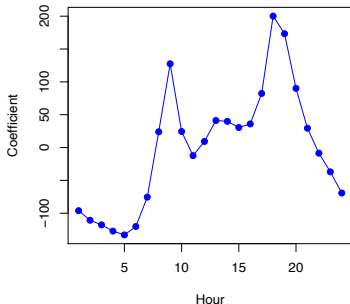
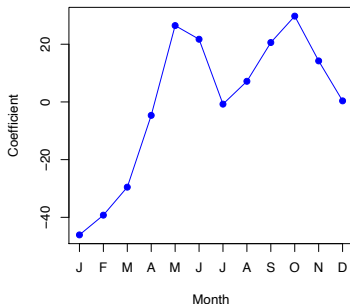
Generalized Linear Models

- Linear regression is used for quantitative responses.
- Linear logistic regression is the counterpart for a binary response, and models the logit of the probability as a linear model.
- Other response types exist, such as non-negative responses, skewed distributions, and more.
- *Generalized linear models* provide a unified framework for dealing with many different response types.

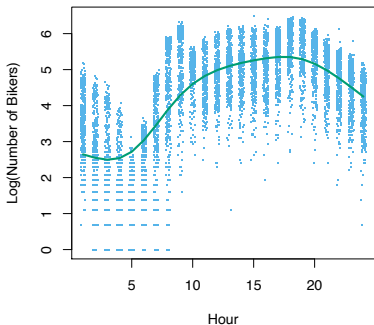
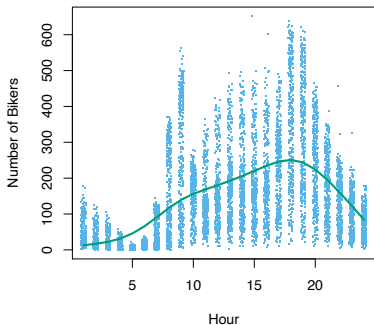
Example: Bikeshare Data

Linear regression with response **bikers**: number of hourly users in bikeshare program in Washington, DC.

	Coefficient	Std. error	z-statistic	p-value
Intercept	73.60	5.13	14.34	0.00
workingday	1.27	1.78	0.71	0.48
temp	157.21	10.26	15.32	0.00
weathersit[cloudy/misty]	-12.89	1.96	-6.56	0.00
weathersit[light rain/snow]	-66.49	2.97	-22.43	0.00
weathersit[heavy rain/snow]	-109.75	76.67	-1.43	0.15

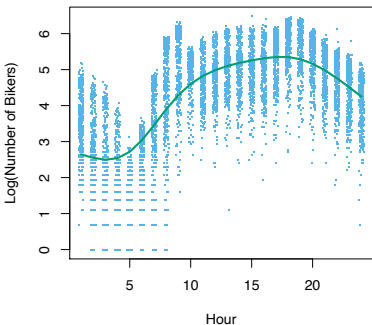
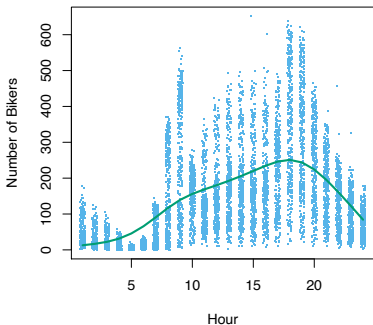


Mean/Variance Relationship



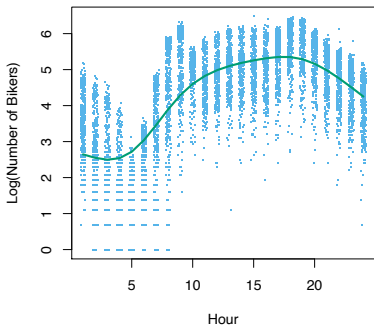
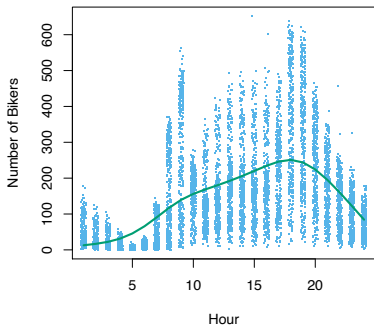
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- 10% of linear model predictions are negative! (not shown here.)
- Taking $\log(\text{bikers})$ alleviates this, but has its own problems: e.g. predictions are on the wrong scale, and some counts are zero!

Poisson Regression Model

- Poisson distribution is useful for modeling counts:

$$\Pr(Y = k) = \frac{e^{-\lambda} \lambda^k}{k!} \text{ for } k = 0, 1, 2, \dots$$

- $\lambda = E(Y) = \text{Var}(Y)$ — i.e. there is a mean/variance dependence.

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- With covariates, we model

$$\log(\lambda(X_1, \dots, X_p)) = \beta_0 + \beta_1 X_1 + \dots + \beta_p X_p$$

or equivalently

$$\lambda(X_1, \dots, X_p) = e^{\beta_0 + \beta_1 X_1 + \dots + \beta_p X_p}.$$

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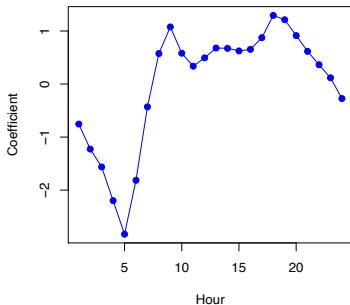
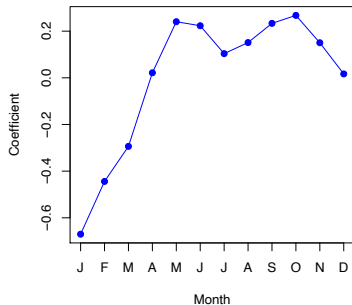
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- Model automatically guarantees that the predictions are non-negative.

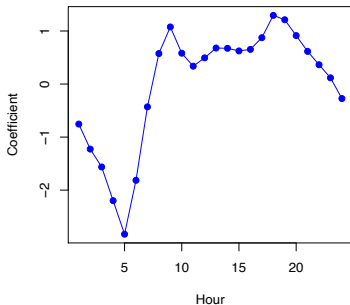
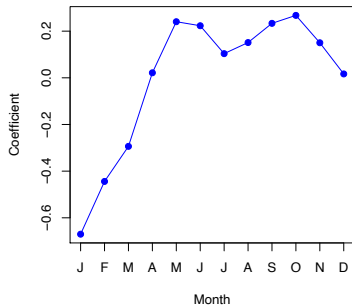
Poisson Regression on Bikeshare Data

	Coefficient	Std. error	z-statistic	p-value
Intercept	4.12	0.01	683.96	0.00
workingday	0.01	0.00	7.5	0.00
temp	0.79	0.01	68.43	0.00
weathersit[cloudy/misty]	-0.08	0.00	-34.53	0.00
weathersit[light rain/snow]	-0.58	0.00	-141.91	0.00
weathersit[heavy rain/snow]	-0.93	0.17	-5.55	0.00



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*In this case the variance is somewhat larger than the mean — a situation known as *overdispersion* — so the p-values are misleadingly small.

Generalized Linear Models

- We have covered three GLMs in this course: Gaussian, binomial and Poisson.
- They each have a characteristic *link* function. This is the transformation of the mean that is represented by a linear model:

$$\eta(\mathbb{E}(Y|X_1, X_2, \dots, X_p)) = \beta_0 + \beta_1 X_1 + \dots + \beta_p X_p.$$

The link functions for linear, logistic and Poisson regression are $\eta(\mu) = \mu$, $\eta(\mu) = \log(\mu/(1 - \mu))$, and $\eta(\mu) = \log(\mu)$, respectively.

- They also each have characteristic *variance* functions.
- The models are fit by maximum-likelihood, and model summaries are produced by `glm()` in R.
- Other GLMS include *Gamma*, *Negative-binomial*, *Inverse Gaussian* and more.